THE MIXED SCHMIDT CONJECTURE IN THE THEORY OF DIOPHANTINE APPROXIMATION

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ABSTRACT. Let $\mathcal{D} = (d_n)_{n=1}^{\infty}$ be a bounded sequence of integers with $d_n \geqslant 2$ and let (i, j) be a pair of strictly positive numbers with $i + j = 1$. We prove that the set of $x \in \mathbb{R}$ for which there exists some constant $c(x) > 0$ such that

$$\max\{|q|_{\mathcal{D}}^{1/i}, \|qx\|^{1/j}\} > c(x)/q \qquad \forall \;\; q \in \mathbb{N}$$

is one quarter winning (in the sense of Schmidt games). Thus the intersection of any countable number of such sets is of full dimension. In turn, this establishes the natural analogue of Schmidt's conjecture within the framework of the de Mathan-Teulié conjecture – also known as the 'Mixed Littlewood Conjecture'.

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1. Introduction

The famous Littlewood conjecture in the theory of simultaneous Diophantine approximation dates back to the 1930's and asserts that for every $(x, y) \in \mathbb{R}^2$, we have that

$$\liminf_{q\to\infty} q\|qx\|\|qy\| = 0. \tag{1}$$

Here and throughout, $\| \, . \, \|$ denotes the distance to the nearest integer. Despite concerted efforts over the years the conjecture remains open. For background and recent 'progress' concerning this fundamental problem see [7, 9] and references within.

The Schmidt conjecture in the theory of simultaneous Diophantine approximation dates back to the 1980's and is linked to Littlewood's conjecture. Given a pair of real numbers i and j such that

$$0 < i, j < 1 \quad \text{and} \quad i + j = 1 \, , \tag{2}$$

let $\mathbf{Bad}(i, j)$ denote the badly approximable set of $(x, y) \in \mathbb{R}^2$ for which there exists a constant $c(x, y) > 0$ such that

$$\max\{ \; \|qx\|^{1/i} \; , \; \|qy\|^{1/j} \, \} \; > \; c(x, y) \; q^{-1} \quad \forall \; q \in \mathbb{N} \; .$$

A consequence of the main result in [1] is the following statement. Throughout, $\dim X$ will denote the Hausdorff dimension of the set X.

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Theorem BPV. *Let (i_t, j_t) be a countable number of pairs of real numbers satisfying* (2). *Suppose that* $\liminf_{t\to\infty} \min\{i_t, j_t\} > 0$. *Then*

$$\dim\Big(\bigcap_{t=1}^{\infty} \mathbf{Bad}(i_t, j_t)\Big) = 2 \ .$$

Thus, the intersection of any finitely many badly approximable sets $\mathbf{Bad}(i, j)$ is trivially non-empty and therefore establishes the following conjecture of Wolfgang M. Schmidt [11]. *For any (i_1, j_1) and (i_2, j_2) satisfying* (2), *we have that*

$$\mathbf{Bad}(i_1, j_1) \cap \mathbf{Bad}(i_2, j_2) \ \neq \ \emptyset \ .$$

To be precise, Schmidt stated the specific problem with $i_1 = j_2 = 1/3$ and $j_1 = i_2 = 2/3$. As noted by Schmidt, a counterexample to his conjecture would imply Littlewood's conjecture. Indeed, the same conclusion is valid if there exists any countable collection of pairs (i_t, j_t) satisfying (2) for which the intersection of the sets $\mathbf{Bad}(i_t, j_t)$ is empty.

Recently, de Mathan and Teulié in [2] proposed the following variant of Littlewood's conjecture. Let $\mathcal{D}$ be a bounded sequence $(d_n)_{n=1}^{\infty}$ of integers greater than or equal to 2 and let

$$D_0 := 1 \quad \text{and} \quad D_n := \prod_{k=1}^{n} d_k \ .$$

Now set

$$\omega_{\mathcal{D}} : \mathbb{N} \to \mathbb{N} : q \mapsto \sup\{n \in \mathbb{N} : q \in D_n\mathbb{Z}\}$$

and

$$|q|_{\mathcal{D}} := 1/D_{\omega_{\mathcal{D}}(q)} = \inf\{1/D_n : q \in D_n\mathbb{Z}\}.$$

When $\mathcal{D}$ is the constant sequence equal to a prime number p, the norm $|\cdot|_{\mathcal{D}}$ is the usual p-adic norm. In analogy with Littlewood's conjecture we have the following statement.

Mixed Littlewood Conjecture. *For every real number x*

$$\liminf_{q\to\infty} q\,|q|_{\mathcal{D}}\,\|qx\| = 0 \,.$$

As with the classical Littlewood conjecture, this attractive problem remains open. The current state of affairs regarding the mixed conjecture is very much comparable to that of the classical one. For background and results related to the mixed Littlewood conjecture see [3–6, 8].

It is somewhat surprising that the analogue of Schmidt's conjecture within the 'mixed' framework has to date escaped attention. The goal of this paper is to investigate such a problem. Given $\mathcal{D}$ as above and a pair of real numbers i and j satisfying (2), let

$$\mathbf{Bad}_{\mathcal{D}}(i, j) := \left\{ x \in \mathbb{R} : \exists\, c(x) > 0 \text{ so that } \max\{|q|_{\mathcal{D}}^{1/i}, ||qx||^{1/j}\} > \frac{c(x)}{q} \quad \forall\, q \in \mathbb{N} \right\} . \qquad (3)$$

A consequence of the Khintchine-type result established in [8] is that $\mathbf{Bad}_{\mathcal{D}}(i, j)$ is of Lebesgue measure zero. The following represents a natural analogue of Schmidt's conjecture.

Mixed Schmidt Conjecture. *For any (i_1, j_1) and (i_2, j_2) satisfying* (2), *we have that*

$$\mathbf{Bad}_{\mathcal{D}}(i_1, j_1) \cap \mathbf{Bad}_{\mathcal{D}}(i_2, j_2) \ \neq \ \emptyset \ .$$

It is easily seen that a counterexample to this conjecture would imply the mixed Littlewood conjecture. Indeed, the same conclusion is valid if there exists any countable collection of pairs (i_t, j_t) satisfying (2) for which the intersection of the sets $\mathbf{Bad}(i_t, j_t)$ is empty. The following constitutes our main result.

Theorem 1. *For any* (i, j) *satisfying* (2)*, the set* $\mathbf{Bad}_{\mathcal{D}}(i, j)$ *is* $1/4$*-winning.*

A consequence of winning is the following full dimension result which settles the mixed Schmidt conjecture. See §2 below for the definition and relevant implications of winning sets.

Theorem 2. *For each* $t \in \mathbb{N}$*, let* $\mathcal{D}_t$ *be a bounded sequence as above and* (i_t, j_t) *be a sequence of pairs of real numbers satisfying* (2)*. Then*

$$\dim \Big(\bigcap_{t=1}^{\infty} \mathbf{Bad}_{\mathcal{D}_t}(i_t, j_t) \Big) = 1 \ .$$

In a nutshell, we are able to establish the mixed analogue of Theorem BPV without the annoying '$\liminf$' assumption.

2. Schmidt games

Wolfgang M. Schmidt introduced the games which now bear his name in [10]. The simplified account which we are about to present is more than adequate for the purposes of this paper.

Suppose that $0 < \alpha < 1$ and $0 < \beta < 1$. Consider the following game involving players A and B. First, B chooses a closed interval $B_1 \subset \mathbb{R}$. Next, A chooses a closed interval A_1 contained in B_1 of length $\alpha|B_1|$. Then, B chooses at will a closed interval B_2 contained in A_1 of length $\beta|A_1|$. Alternating in this manner between the two players, generates a nested sequence of closed intervals in $\mathbb{R}$:

$$B_1 \supset A_1 \supset B_2 \supset A_2 \supset \ldots \supset B_m \supset A_m \supset \ldots$$

with lengths

$$|B_m| \;=\; (\alpha\,\beta)^{m-1}|B_1| \quad \text{and} \quad |A_m| \;=\; \alpha\,|B_m|\,.$$

A subset S of $\mathbb{R}$ is said to be (α, β)*-winning* if A can play in such a way that the unique point of intersection

$$\bigcap_{m=1}^{\infty} B_m \;=\; \bigcap_{m=1}^{\infty} A_m$$

lies in S, regardless of how B plays. The set S is called α*-winning* if it is (α, β)-winning for all $\beta \in (0, 1)$. Finally, S is simply called *winning* if it is α-winning for some α. Informally, player B tries to stay away from the 'target' set S whilst player A tries to land on S. The following results are due to Schmidt [10].

Theorem S1. *If* $S \subset \mathbb{R}$ *is an* α*-winning set, then* $\dim S = 1$.

Theorem S2. *The intersection of countably many* α*-winning sets is* α*-winning.*

Armed with these statements it is obvious that

$$\text{Theorem 1} \quad \Longrightarrow \quad \text{Theorem } \ 2 \ .$$

3. Proof of Theorem 1

For any real $c > 0$, let $\mathbf{Bad}_{\mathcal{D}}(c; i, j)$ be the set of $x \in \mathbb{R}$ such that

$$\max\{|q|_{\mathcal{D}}^{1/i}, ||qx||^{1/j}\} > \frac{c}{q} \qquad \forall\ q \in \mathbb{N}. \tag{4}$$

It is easily seen that $\mathbf{Bad}_{\mathcal{D}}(c; i, j)$ is a subset of $\mathbf{Bad}_{\mathcal{D}}(i, j)$. Moreover, it has a natural geometric interpretation in terms of avoiding neighbourhoods of rational numbers. Let

$$\mathcal{C}_c := \{r/q \in \mathbb{Q} \,:\, (r,q) = 1\,, q > 0 \ \text{ and } \ |q|_{\mathcal{D}} < c^i q^{-i}\}$$

and

$$\Delta_c(r/q) := \left[\frac{r}{q} - \frac{c^j}{q^{1+j}}\,,\ \frac{r}{q} + \frac{c^j}{q^{1+j}}\right]$$

Then,

$$\mathbf{Bad}_{\mathcal{D}}(c; i, j) = \{x \in \mathbb{R} : x \notin \Delta_c(r/q) \ \ \forall\ r/q \in \mathcal{C}_c\}\ .$$

Now with reference to §2, let $\mathbf{Bad}_{\mathcal{D}}(i, j)$ be the target set S and $\alpha \in (0, 1)$ be a fixed real number at our disposal. Suppose player B has chosen some $\beta \in (0, 1)$ and an interval B_1. Let

$$R := \frac{1}{\alpha\beta} > 1$$

and fix $c > 0$ such that

$$c \,<\, \min\left\{(4\,R\,|B_1|^{-1})^{-1/j}\,,\, (2\,R^{\frac{i}{j+1}}\,|B_1|)^{-1/i}\right\}. \tag{5}$$

By definition, for each $m \geq 1$

$$|B_m| \,=\, R^{-m+1}|B_1|\,.$$

The 'winning' strategy that player A adopts is as follows. *If B_m is the interval player A inherits from player B, then A will choose an interval $A_m \subset B_m$ with $|A_m| = \alpha\,|B_m|$ such that*

$$A_m \cap \Delta_c(r/q) = \emptyset \quad \forall\ r/q \in \mathcal{C}_c \quad \textit{with} \ \ 0 < q^{1+j} < R^{m-1}\ . \tag{6}$$

Suppose for the moment that player A can adopt this strategy with $\alpha = 1/4$. Then

$$\bigcap_{m=1}^{\infty} A_m \,\in \mathbf{Bad}_{\mathcal{D}}(c; i, j) \ \subset\ \mathbf{Bad}_{\mathcal{D}}(i, j)$$

and it follows that $\mathbf{Bad}_{\mathcal{D}}(i, j)$ is 1/4–winning as claimed. We use induction to prove that such a strategy exists.

For $m = 1$, player A can trivially choose an interval A_1 satisfying (6) since there are no rationals with $0 < q < 1$. Now suppose the intervals

$$B_1 \supset A_1 \supset B_2 \supset A_2 \supset \ldots \supset B_n \supset A_n \supset B_{n+1}$$

have been determined with each A_m $(1 \leqslant m \leqslant n)$ satisfying (6). The goal is to show that there exists an interval A_{n+1} satisfying (6). To begin with observe that since B_{n+1} is nested in A_n, we have that

$$B_{n+1} \cap \Delta_c(r/q) = \emptyset \ \ \forall\ r/q \in \mathcal{C}_c \ \text{ with } \ 0 < q^{1+j} < R^{n-1}.$$

Thus, since A_{n+1} is to be nested in B_{n+1}, it follows that A_{n+1} will satisfy (6) if

$$A_{n+1} \cap \Delta_c(r/q) = \emptyset \quad \forall \; r/q \in \mathcal{C}_c(n) \;, \tag{7}$$

where

$$\mathcal{C}_c(n) := \{ r/q \in \mathcal{C}_c \; : \; R^{n-1} \leqslant q^{1+j} < R^n \}.$$

Fact 1. Let $r/q \in \mathcal{C}_c(n)$. Then

$$\begin{aligned} |\Delta_c(r/q)| &= \frac{2c^j}{q^{1+j}} \leqslant 2c^j \, R^{-n+1} \\ &\stackrel{(5)}{<} \tfrac{1}{2} \, |B_{n+1}| \, . \end{aligned}$$

Fact 2. Let $r_1/q_1, r_2/q_2 \in \mathcal{C}_c(n)$. Then there exists non-negative integers k_1 and k_2 such that

$$q_s = D_{k_s} q_s^* \quad \text{and} \quad q_s \notin D_{k_s+1} \, \mathbb{Z} \qquad (s = 1, 2) \, .$$

Since $|q_s|_{\mathcal{D}} < c^i \, q_s^{-i}$, we have that

$$D_{k_s} > c^{-i} \, q_s^i \geqslant c^{-i} \, R^{\frac{(n-1)i}{j+1}} \, .$$

Hence, it follows that

$$(q_1, q_2) > c^{-i} R^{\frac{(n-1)i}{j+1}}$$

and so

$$\begin{aligned} \left| \frac{r_1}{q_1} - \frac{r_2}{q_2} \right| &\geqslant \frac{(q_1, q_2)}{q_1 q_2} > c^{-i} R^{\frac{(n-1)i}{j+1}} R^{-\frac{2n}{j+1}} \\ &= c^{-i} \, R^{-\frac{i}{j+1}} \, R^{-n} \\ &\stackrel{(5)}{>} 2|B_{n+1}| \, . \end{aligned}$$

A straightforward consequence of the above two facts is that there is at most one rational $r/q \in \mathcal{C}_c(n)$ such that

$$\Delta(r/q) \, \cap \, B_{n+1} \neq \emptyset \, .$$

This together with Fact 1 implies that there is at least one interval $I \subset B_{n+1}$ of length $\frac{1}{4}|B_{n+1}|$ that avoids $\Delta_c(r/q)$ for all $r/q \in \mathcal{C}_c(n)$. With $\alpha = 1/4$, player A takes A_{n+1} to be any such interval and this completes the induction step. The upshot is that for $\alpha = 1/4$ and any $\beta \in (0, 1)$ there exists a winning strategy for player A.

Remark. The arguments used to prove Theorem 1 can be naturally modified to establish the following statement. For any given finite number of sequences $\mathcal{D}_1, \ldots, \mathcal{D}_s$ and strictly positive real numbers $i_1, \ldots, i_s, j$ satisfying $i_1 + \ldots + i_s + j = 1$, the set of $x \in \mathbb{R}$ such that

$$\max\{ \, |q|_{\mathcal{D}_1}^{1/i_1}, |q|_{\mathcal{D}_2}^{1/i_2}, \ldots, |q|_{\mathcal{D}_s}^{1/i_s}, ||qx||^{1/j} \, \} > c(x) \; q^{-1} \quad \forall \; q \in \mathbb{N}$$

is 1/4–winning.

4. The genuine mixed Schmidt conjecture?

If $s = 0$, then let us adopt the convention that $x^{1/s} := 0$. Then $\mathbf{Bad}_{\mathcal{D}}(0,1)$ is identified with the standard set $\mathbf{Bad}$ of badly approximable numbers and $\mathbf{Bad}_{\mathcal{D}}(1,0)$ is identified with $\mathbb{R}$. With this in mind, we are able to replace '$<$' by '$\leqslant$' in (2) without effecting the statements of Theorems 1 & 2. Moreover, it enables us to consider the following generalization of Schmidt's conjecture. Given $\mathcal{D}$ and real numbers i, j, k satisfying

$$0 \leqslant i, j, k \leqslant 1 \quad \text{and} \quad i + j + k = 1\,, \tag{8}$$

let $\mathbf{Bad}_{\mathcal{D}}(i,j,k)$ denote the set of $(x,y) \in \mathbb{R}^2$ for which there exists a constant $c(x,y) > 0$ such that

$$\max\{\, |q|_{\mathcal{D}}^{1/i} \ , \ \|qx\|^{1/j} \ , \ \|qy\|^{1/k} \,\} \ > \ c(x,y)\ q^{-1} \quad \forall\, q \in \mathbb{N}\ .$$

Naturally, $\mathbf{Bad}_{\mathcal{D}}(1,0,0) := \mathbb{R}^2$, $\mathbf{Bad}_{\mathcal{D}}(0,1,0) := \mathbf{Bad} \times \mathbb{R}$, $\mathbf{Bad}_{\mathcal{D}}(0,0,1) := \mathbb{R} \times \mathbf{Bad}$, $\mathbf{Bad}_{\mathcal{D}}(0,j,k) := \mathbf{Bad}(j,k)$, $\mathbf{Bad}_{\mathcal{D}}(i,0,k) := \mathbf{Bad}_{\mathcal{D}}(i,k)$ and $\mathbf{Bad}_{\mathcal{D}}(i,j,0) := \mathbf{Bad}_{\mathcal{D}}(i,j)$.

Conjecture 1. *For any (i_1, j_1, k_1) and (i_2, j_2, k_1) satisfying (8), we have that*

$$\mathbf{Bad}_{\mathcal{D}}(i_1,j_1,k_1) \cap \mathbf{Bad}_{\mathcal{D}}(i_2,j_2,k_2) \ \neq\ \emptyset\ .$$

Observe that when $i_1 = i_2 = 0$, this 'mixed' conjecture reduces to the classical Schmidt conjecture. On the other hand, when $j_1 = j_2 = 0$ or $k_1 = k_2 = 0$ the above conjecture reduces to the mixed Schmidt conjecture investigated in this paper. In view of the results established to date it is reasonable to expect that the following is true.

Conjecture 2. *Let (i_t, j_t, k_t) be a countable number of triples of real numbers satisfying (2). Then*

$$\dim\Big(\bigcap_{t=1}^{\infty} \mathbf{Bad}_{\mathcal{D}}(i_t, j_t, k_t)\Big) = 2\ .$$

Note that this conjecture is open even when $i_t = 0$ for all $t \in \mathbb{N}$. The point is that this situation is not covered by Theorem BPV since we have not imposed the condition that $\liminf_{t\to\infty} \min\{j_t, k_t\} > 0$.

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